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Cycles and circles in roundoff errors

George G. Szpiro*
Institute for Empirical Research in Economics, University of Zürich, Zürich, Switzerland
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When a series of measurements is performed with increasingly coarse (or increasingly fine) precision, consecutive observations seem to be erratically distributed at first, and then organize themselves into cycles and patterns. The patterns, which arise because of roundoff errors, are related to a notion in number theory, the so-called Farey sequence.

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I. INTRODUCTION

A strange phenomenon may be observed when consecutive measurements of dilating or contracting objects are made. Due to the finite precision of the measurements, roundoff errors arise, which first lead to seemingly random observations, and then organize themselves into a series of dome-shaped patterns. These patterns can be explained in the context of a variant of circle maps, which can, in turn, be understood by employing number theory. We show that the patterns are closely associated with the so-called Farey sequence, i.e., the succession of rational numbers on the real line. The phenomenon has been observed in such diverse areas as the flow of traffic [1] (without actually being recognized), and production levels in economics [2].

II. A MEASUREMENT PROBLEM IN RELATIVISTIC DYNAMICS

We illustrate the topic by describing measurements of an object traveling near the speed of light c. Consider a measuring rod at rest, which is calibrated into G units of length u. A second rod travels parallel to the measuring rod at a velocity v. The length of the traveling rod is measured by observing how many units on the measuring rod make up the length of the traveling rod, and converting the observed length to the length L at rest. Hence, the length is

$$L=G\gamma\left[\frac{u}{\gamma}\right], \tag{1}$$

where γ is the Lorentz factor for the transformation of lengths, $(1-v^2/c^2)^{0.5}$ and $G\gamma$ is the number of units. Assume that the rod accelerates, and its length is measured at regular intervals in time [3]. If measurements could be performed with infinite precision, the observed length L would be identical at each measurement, regardless of the velocity with which the rod travels. However, since measurements can only be performed with finite precision, the number of units is rounded to the nearest integer. Hence the length measured at time t, L_t, becomes

$$L_t=\rho(G\gamma_t)\left[\frac{u}{\gamma_t}\right], \tag{2}$$

where $\rho(\ )$ designates the rounded number. To simplify the following exposition, we assume that both rods at rest are of unit length (hence $u=1/G$). We obtain

$$L_t=\rho(G\gamma_t)\frac{u}{\gamma_t}=\rho(G\gamma_t)\frac{1}{G\gamma_t}. \tag{3}$$

Now, also to simplify exposition, assume acceleration such that

$$v_t=c\left[1-\frac{1}{t^2}\right]^{0.5}, \tag{4}$$

and Eq. (3) becomes

$$L_t=\rho\left[\frac{G}{t}\right]\frac{t}{G}. \tag{5}$$

Using $G=10^6$, Fig. 1 depicts a series of measurements for $t=1,2,\ldots$. As could be expected, the higher the velocity, the less accurate the measurements become. But another, more surprising, phenomenon, emerges: the measured lengths exhibit seemingly random behavior at first [Fig. 1(a)], while, for higher values of t, measurements form patterns of domes [Fig. 1(b)].

III. ROUNDOFF ERRORS

In order to analyze this phenomenon, Eq. (5) is rewritten as

$$L_t=\left[\rho\left[\frac{G}{t}\right]-\left[\frac{G}{t}\right]\right]\left[\frac{t}{G}\right]+1. \tag{6}$$

The inner expression,

$$\epsilon_t=\rho\left[\frac{G}{t}\right]-\left[\frac{G}{t}\right], \tag{7}$$

is the normalized roundoff error of the measurements, and we do not restrict generality by limiting the analysis to the series given by Eq. (7). In order to examine the occurrence of patterns in Fig. 1(b), we first compute the step size between consecutive observations,

$$W_t=\epsilon_{t+1}-\epsilon_t=\left[\frac{G}{t^2+t}\right]-\rho\left[\frac{G}{t}\right]+\rho\left[\frac{G}{t+1}\right]. \tag{8}$$

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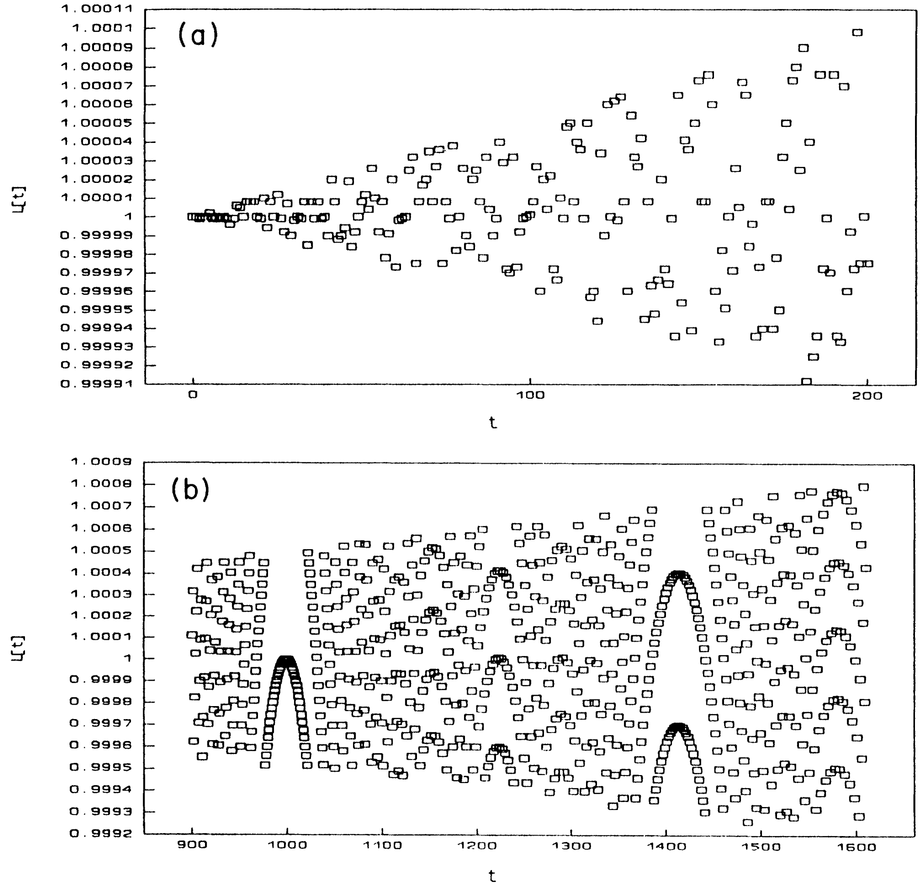

FIG. 1. (a) Relativistic measurements with finite precision. (b) Patterns produced by round-off errors.

The new series W_t takes on rational values between $+1$ and -1. Since after every few steps in one direction, the observations restart at the other end, we have positive steps interspersed with negative ones, and vice versa. (For example, if $W_t = W_{t+1} = W_{t+2} = +0.3$, then $W_{t+3} = -0.7$.) Hence, the points of the series are arranged in two piecewise, monotonically decreasing strands which are offset from one another by a vertical distance of 1. (See Fig. 2.) While one strand decreases from 0 to -1, the other decreases from $+1$ to 0 on intervals I_K, which reach from whenever $G/(t^2+t)$ is an integer to the next such occurrence

$$I_K = \left[t \,\middle|\, K+1 > \frac{G}{t^2+t} > K \right], \quad K = 1,2,3,\ldots . \tag{9}$$

Hence, the approximate end points of each interval I_K are

$$t_K = \left[\frac{G}{K} \right]^{0.5} . \tag{10}$$

For G equal to, say, 10^6 some of the end points are $t_4 = 500$, $t_3 = 577$, $t_2 = 707$, and, finally, $t_1 = 1000$. Starting with high values of K, the intervals are narrow at first, and then increase in width, the last one ranging from 1000 to infinity. The series W_t is therefore approximately periodic in $1/t^2$.

IV. CIRCLE MAPS

From Eq. (8) we have

$$\epsilon_{t+1} = \epsilon_t + W(t) , \tag{11}$$

which remotely resembles a circle map: the term $W(t)$ calls to mind the winding number [4]. However, since $W(t)$ changes with time, we will call it the "momentary winding number at time t" (MWN). By definition $W(t)$ moves from one rational number to another.

Assume $W(t^*) = \alpha/\beta$, where α and β are relatively prime. In principle, as long as $W(t^*+q)$ equals α/β or $\alpha/\beta \pm 1$ ($q = 1,2,3,\ldots$), a cycle of length β appears, since $\epsilon_{t+\beta} = \epsilon_t$ (plus integer). However, since the MWN changes continuously, we only have an approximate cycle which persists for as long as the two strands of $W(t)$ are not too far removed from α/β and $\alpha/\beta \pm 1$, respectively. Moreover, since the strands decrease, we have $\Delta^2 \epsilon_t < 0$ (where Δ^2 denotes the second difference), and the juxtaposition of approximate cycles creates the characteristic domes. Their multiplicity is given by the MWN's denominator β. For example, for $W(t) = \frac{1}{2}$ (which occurs at $t = 1414, 816, 632, \ldots$) we observe "double" domes, while domes of multiplicity three are observed for $W(t) = \frac{1}{3}$ (which occurs at $1732, 866, 655, \ldots$), and for $W(t) = \frac{2}{3}$ (at $t = 1224, 775, 612, \ldots$).

When $W(t)$ passes from one rational number, α_1/β_1, to

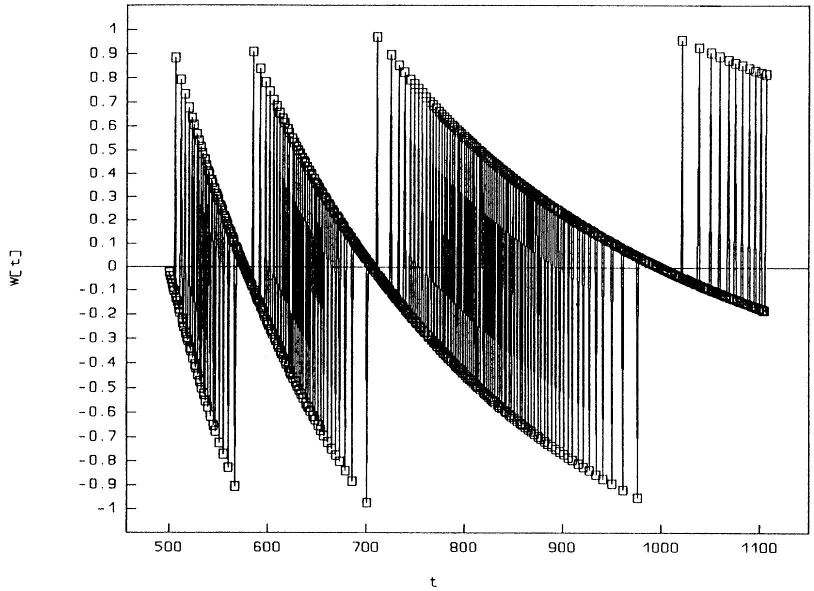

FIG. 2. Momentary winding numbers.

another, α_2/β_2, the domes of multiplicity β_1 melt into domes of multiplicity β_2. If the interval I_K is too narrow, however, as is the case for large values of K (low values of t), the domes melt away before they can fully develop, and the observations seem to be erratically distributed [as in Fig. 1(a)].

The above implies that the sequencing of domes follows the sequencing of the rational numbers on the unit interval, i.e., to the so-called Farey sequence [5]. However, since not all rational numbers are visited by $W(t)$, some domes are left out. Other domes are omitted because the MWN's denominator is too high: for a certain MWN, say $w^*=\alpha/\beta$, $W(t)$ must stay close to w^* (or $w^*\pm 1$) for at least β periods in order to complete one cycle, and at least four or five cycles are needed to identify a dome. If β is higher than about 10, it is unlikely that $W(t)$ stays close to w^* (or $w^*\pm 1$) for that long, since $W(t)$ continuously changes. Other than that, all the properties of Faray sequences hold for the order in which the domes appear. [For example, the multiplicity of the largest dome in between two larger domes A and B is equal to the sum of the multiplicities of domes A and B. See Fig. 1(b).]

The width of the domes, that is, the time period for which the approximate cycles persist, is determined by two factors: (a) by the time it takes for a dome belonging to the MWN α_1/β_1 to melt into the dome belonging to MWN α_2/β_2 (i.e., for a dome of multiplicity β_1 to melt into a dome of multiplicity β_2). This time is related to the distance between "adjacent" rational numbers, where adjacent is defined in the subset of rationals whose denominator is less than or equal to n (for the reason given above, n is about 10). For this subset of rational numbers the distance tends to be larger, the lower the denominator. (The reason is that the distance between two adjacent rational numbers, a/b and c/d, is $1/bd$, which is bounded from below by $1/bn$ [5]. This lower bound is larger the smaller b.) (b) Since the strands of $W(t)$ decrease at a decreasing rate $[\Delta^2 W(t)<0]$, small MWN's melt away faster than large ones. Hence the width of the domes increases with t, and for low values of t we obtain the seemingly random behavior of Fig. 1(a).

V. SUMMARY AND CONCLUSIONS

The seemingly random observations which appear when measurements of lengths are performed with increasingly coarse (or increasingly fine) instruments, and which then organize themselves into cycles and patterns, are engendered by the roundoff errors in the measurement process. We show that the series of errors is produced by a variant of the circle map, where the winding number, which continually changes, is suitably redefined. The patterns are related to the Farey sequence, a concept in number theory which describes the sequencing of rational numbers on the real line.

In this paper the problem was motivated by relativistic measurements, but it also appears in other circumstances, and even in the social sciences. In economics, for example, where the value of an asset is measured in dollars and cents, increasing inflation causes the units of measurements to become successively finer.

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*Address for correspondence: P.O. Box 6278, Jerusalem, Israel. Fax: 972-2-24 48 36. Electronic mail: NZZJRS@DM.RS.CH

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[3] The problem is suggestive of Mandelbrot's way of introducing fractals, by imagining measurements of a

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[4] See, for example, P. Bak, T. Bohr, and M. H. Jensen, in *Directions in Chaos,* edited by Hao Bai-Lin (World Scientific, Singapore, 1988), Vol. 2, pp. 16–45.

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